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STUDY ON GEOMETRIC BROWNIAN MOTION WITH APPLICATIONS

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ABSTRACT

The purpose of this paper is to give a detail study on geometric Brownian motion with some applications.

Keywords- *Geometric Brownian motion, Ito lemma, delays geometric Brownian motion, geometric Brownian with jump.*

I. INTRODUCTION

A random differential equation is a differential equation in which one or more of the terms is a random process resulting in a solution which is itself a random process. The existence and uniqueness of solution for the random differential equation is found in [1]. A geometric Brownian motion (GBM for briefly) is an important example of random processes satisfying a random differential equation and play great role in mathematical finance as it use for modeling values of investments or stock prices in the Black–Scholes model, GBM is widely used model of stock price behavior ,see[2]. The economist prefers geometric Brownian motion model for market prices as it is everywhere positive with probability 1.

Let Ω be the set of all possible outcomes of any random experiment , throughout this study we will denote by X_t to the continuous time random process defined on the filtered probability space $\left(\Omega, F, \{F_t\}_{t\in T}, P\right)$ where, F is the σ-algebra of events, $\{F_t\}_{t\in T}$ denotes the information generated by the process X_t over time interval $[0,T]$ and P is the probability measure . Furthermore, the process X_t is adapted to the filtration $\{F_t\}_{t\in T}$.

Definition: A random variable X have the lognormal distribution (with parameters μ and σ) if log (X) is normally distributed (log(X) ~ N(μ,σ^2)).

Consider the complete filtered probability space $\left(\Omega, F, \{F\}_{t>o}, P\right)$ that satisfy the usual conditions.

Definition: On the time interval $[0,\infty]$, a real-valued random process $W_t = W(t,\omega)$ is Brownian motion (wiener process) if it satisfies the following properties:

1) $W_0 = 0$
2) For $0 \le s < t \le T$, $W_t - W_s$ has normal distribution $N\,(0, t-s)$ with mean zero and variance $t-s$
3) For $0 \le s' < t' < s < t \le T$, $W_t - W_s$ independent of $W_{t'} - W_{s'}$

1-dimensional Itô formula:

Let X_t be 1- dimensional Itô processes given by $d\,X_t = u(t)\,dt + v(t)\,dB_t$

Let $g(x,t) \in C^2\left([0,\infty) \times R\right)$, then $Y_t = g\left(t, X_t\right)$ is again Itô processes and

$$d\,Y_t = \frac{\partial g\left(t, X_t\right)}{\partial t} dt + \frac{\partial g\left(t, X_t\right)}{\partial x} d\,X_t + \frac{1}{2}\frac{\partial^2 g\left(t, X_t\right)}{\partial x^2}\left(d\,X_t\right)^2$$

Where $\left(d\,X_t\right)^2 = \left(d\,X_t\right).\left(d\,X_t\right)$ is computed according to the following rules

$$dt.dt = dB_t.dt = dB_t.dt = 0 \ , \ dB_t.\, dB_t = (dB_t)^2 = dt \ , \ dB_i\, dB_j = 0 \ , \forall\ i \neq j \quad (1)$$

(1) also known as Ito multiplication table .For the proof see [1]

Ito formula is fundamental theorem for evaluating Ito integral.

Multi-dimensional Itô process:

Let $B(t,\omega) = \left(B_1(t,\omega)\ , B_2(t,\omega)\ , \cdots B_m(t,\omega)\right)$, denote to *m*- dimensional Brownian motion , for each $U_i(t,\omega)$, $1 \le i \le n$, $V_{ij}\ (t,\omega)$, $1 \le j \le m$, the multi-dimensional Itô process defined as : $dX_t = U_i\, dt + V_i(t)\ d\, B_i\ (t)$

X_t , U_i , V_i and B_t defined in matrix and vector form as:

$$X_t = \begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \\ \vdots \\ X_n(t) \end{pmatrix} , U_i = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{pmatrix} , V_{ij} = \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1m} \\ v_{21} & v_{22} & \cdots & v_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nm} \end{pmatrix} , B_t = \begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ \vdots \\ B_m(t) \end{pmatrix}$$

Theorem: Multi-dimensional Itô formula:

Consider the multi-dimensional Ito process $dX_t = U_i\, dt + V_i(t)\ d\, B_i\ (t)$, where the vector $U_i = \left(U_1\ , U_2\ , U_3 \cdots U_n\right)$ and the matrix $V_{ij} = \left(V_{11}\ , V_{12}\ , \cdots , V_{1d}\right)$ have L_2 components and B_t is the vector of d independent Brownian motions.

Let $g\left(t\ ,\ X_t\right)$ be twice continuously differentiable function from R^d into R then $Y_t = g\left(t\ ,\ X_t\right)$ is also an Ito process and

$$d\ Y_k = \frac{\partial g_k\left(t, X_t\right)}{\partial t}\, dt + \sum_{i=1}^{d} \frac{\partial g_k\left(t, X_t\right)}{\partial x_i}\, dX_i + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 g_k\left(t, X_t\right)}{\partial x_i\, \partial x_j}\, dX_i\, dX_j$$

Where $dX_i\ dX_j$ is computed by using the Ito multiplication rule in (1)

Definition: A geometric Brownian motion is a continuous-time random process , and we say that the random process S_t fellow the geometric Brownian motion if it satisfy the following random differential equation

$$dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dW_t \quad (2)$$

where μ represents the constant drift or trend of the process , σ represents the amount of random variation around the trend and W_t is a Wiener process or Brownian motion.

II. SOLUTION OF THE GEOMETRIC BROWNIAN MOTION

Denote by $S_0 = S(0)$, to the initial value which is $F_0 -$ measurable, then under Ito interpretation, equation (2) has the analytic solution:

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right) \quad (3)$$

Which is log-normally distributed random variable with expected value $E(S_t) = S_0 e^{\mu t}$ and variance $Var(S_t) = S_0^2 e^{2\mu t} (e^{\sigma^2 t} - 1)$.

Geometric Brownian motion is that it is Markovian since it satisfies the Markov chain property, thus the future states $S(t+h)$ depends only on the future increment of the Brownian motion, see [4].

Definition: A random process X_t is said to be F_t −martingale if the following conditions are satisfied:

1- X_t is adapted to the filtration $\{F_t\}_{t \in T}$
2- $\forall \ t \geq 0$, $E[X_t] < \infty$
3- For every s and t such that $0 < s < t$, $E[X_t | F_s] = X_s$

Condition (3) means that the information in the σ-algebra F_s is quiet enough to determine the value of X_s, It is easy to prove that geometric Brownian motion is martingale.

III. MULTIDIMENSIONAL GEOMETRIC BROWNIAN MOTION

The GBM (2) can be extending to multidimensional case as fellow:

$$d S_i(t) = \mu_i S_i(t)\, dt + S_i(t) \sum_{j=1}^{n} \sigma_{i,j} \, dW_j(t) \ , \ i = 1,2,3\cdots,n \quad (4)$$

Where $S_i(t)$, μ_i , , $\sigma_{i,j}$ and $W_j(t)$ defined in vector and matrix notation as fellow:

$$S_i(t) = \begin{pmatrix} S_t^1 \\ S_t^2 \\ S_t^2 \\ \vdots \\ S_t^m \end{pmatrix} \quad \mu_i = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_m \end{pmatrix} \quad \sigma_{i.j} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1k} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mk} \end{pmatrix}, \ W_j(t) = \begin{pmatrix} W_t^1 \\ W_t^2 \\ W_t^2 \\ \vdots \\ W_t^m \end{pmatrix}$$

If $S_i(0)$ is the initial value, then by using Itô's lemma the general solution of (4) is:

$$S_i(t) = S_i(0) \ \exp\left(\left(\mu_i - \frac{1}{2}\sum_{j=1}^{n} \sigma_{i,j}{}^2\right) t + \sum_{j=1}^{n} \sigma_{i,j} \, W_i(t)\right)$$

Càdlàg function: A function $f : E \to M$ is called a càdlàg function if for every $t \in E$ the function $f(t)$ is right-continuous with left limits.

If the sample path of the random process X_t is right continuous with left limit, we set $X_{t-} = \lim_{s \to t, s < t} X_s$ and $\Delta X_t = X_t - X_{t-}$, ΔX_t represent the size of jump of the process X_t at time t , if there is no jump then ΔX is zero, see[6].

Definition: A jump process is type of random process that has discrete movements called jumps, the very useful jump process is the standard Poisson process N_t with jumps of size +1 only, that is $\left(\Delta N_t = N_t - N_{t-} = 1\right)$, and whose paths are constant in between two jumps. If N_t jumps occur in $(0, t]$,then for some λ_t and for each $s \geq 0$ and $t > 0$, $N_{s+t} - N_s \sim$ Poisson with mean λ_t .

Definition: The process

$$Y_t = \sum_{k=1}^{N_t} Z_k \quad , t \in R \tag{5}$$

Is called a compound Poisson process.

Random Integrals with Jumps: Given the random process X_t we let the random integral of X_t with respect Y_t defined by $\int_0^T X_t \, d Y_t = \sum_{i=1}^{N_i} X_{t_k} Z_k$

In particular the compound Poisson process Y_t in (5) admits the random integral representation

$$Y_t = Y_0 + \int_0^t Z_{N_s} \, d N_s \tag{6}$$

Ito formula with jump: consider the case of a standard Poisson process N_t with intensity λ_t and N_{s-} be the left limit of the Poisson process at time *s,* then

$$\begin{aligned} f(N_t) - f(0) &= \sum_{k=1}^{N_t} f(k) - f(k-1) \\ &= \sum_{k=1}^{N_t} f\left(1 + N_{s-}\right) - f\left(N_{s-}\right) d N_s \\ &= \sum_{k=1}^{N_t} f\left(N_s\right) - f\left(N_{s-}\right) d N_s \end{aligned}$$

For the compound Poisson process Y_t

$$\begin{aligned} f(Y_t) - f(0) &= \sum_{k=1}^{N_t} f\left(Y_{T_k} + Z_k\right) - f\left(Y_{T_k -}\right) \\ &= \sum_{k=1}^{N_t} f\left(Y_{s-} + Z_{N_s}\right) - f\left(Y_{s-}\right) d N_s \\ &= \sum_{k=1}^{N_t} f\left(Y_s\right) - f\left(Y_{s-}\right) d N_s \end{aligned}$$

And for the process

$$d Y_i = u \, dt + v \, dW_t + \delta \, d N_t \tag{7}$$

Where δ is the fraction rate.

And if $f(t) \in C^2\left([0, \infty) \times R\right)$, the process $Y_t = f(t)$ is again Ito process and

$$f(Y_i) = f(Y_0) + \int_0^t u\, f'(Y_s)\, dW_s + \frac{1}{2}\int_0^t u^2 f''(Y_s)\, dW_s + \int_0^t v\, f'(Y_s)\, ds$$

$$+ \int_0^t \left(f(Y_s) - f(Y_{s-})\right) dN_s \quad (8)$$

The Ito formula for jump processes used the following extension Ito multiplication table:

$\cdot$	dt	dB_t	dN_t
dt	0	0	0
dB_t	0	dt	0
dN_t	0	0	dN_t

see[7]

IV. GEOMETRIC BROWNIAN MOTION WITH JUMPS

Generally the random differential equations with jump component has great importance in different areas of science. The problem of this kind of processes with jumps is considered in [1], [3]. Geometric Brownian motion with Jumps play essential role in financial market when the stock prices, and prices of other assets show jumps which usually caused by unpredictable events or sudden shift.

Consider that an asset price S_t with the percentage drift μ and the percentage volatility σ follow GBM, but each jump reduces the asset price by a fraction δ, then S_t should satisfy a differential equation like

$$dS(t) = \mu\, S(t)\, dt + \sigma\, S(t)\, dW(t) - S(t)\, \delta\, dN_t \quad (9)$$

Such equation is known as geometric Brownian motion with jump.

Through integration we come to random integral equation

$$S(t) = S(0) + \int_0^t \mu\, S(u)\, du + \int_0^t \sigma\, S(u)\, dW(u) - \int_0^t S(u)\, \delta\, dN_u \quad (10)$$

Where $S(0)$ is the initial value. Again under Itô's interpretation (8) , the general solution of (9) is:

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)(1-\delta)^{N_t} \quad (11)$$

V. DELAY GEOMETRIC BROWNIAN MOTION

Delay differential equations (DDEs) are a type of differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times.

Delay geometric Brownian motion (DGBMs for briefly) or geometric Brownian motion with delay is the solution of a random differential equation where the drift and diffusion coefficient depend linearly on the past of the solution.

Let $S(t)$ be the price of risky asset at time t , and assume that $S(t)$ is the geometric Brownian motions that is:

$$dS(t) = r\, S(t)\, dt + V(t)\, S(t)\, dW(t) \quad (12)$$

where $r > 0$ is the risk-free interest rate , $V(t)$ represents the volatility and W_t is a Wiener process or Brownian motion

The case when the volatility $V(t)$ can be represented as a function of $S(t-\tau)$, say $V\left(S(t-\tau)\right)$, equation (12) becomes:

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$$d\,S(t) = r\,S(t)\,dt + V\big(S(t-\tau)\big)\,S(t)\,dW(t) \tag{13}$$

As the option price at time $t-\tau$ is clearly dependent on the underlying asset price $S(t-\tau)$.

In the other case if the volatility is a function of the past states $S(t-\tau_1)$, $S(t-\tau_2)$, $\cdots$, $S(t-\tau_n)$, then $S(t)$ satisfy:

$$d\,S(t) = r\,S(t)\,dt + V\big(S(t-\tau_1), S(t-\tau_2), \cdots, S(t-\tau_n)\big)\,S(t)\,dW(t) \tag{14}$$

Such equations ((13) and (14)) are delay geometric Brownian motions see[5].

Applications: Geometric Brownian motion has many applications in real life, for example in mathematical finance and economics as we mentioned above the geometric Brownian motion is essential model for the stock prices.

As any other differential equation in describing natural phenomenon geometric Brownian motion model is used to study and describe the growth of populations as it consider the random influences (noise element) on the growth process which are neglected in deterministic approach .

VI. GEOMETRIC BROWNIAN MOTION AS POPULATION GROWTH MODEL

Consider the simple (deterministic) population growth model

$$\frac{d\,N(t)}{dt} = a(t)\;N(t)\;,\; N(0) = N_0 \text{ (costant)} \tag{14}$$

where $N(t)$ is the size of population at time t and $a(t)$ is the relative rate of growth at time t, for some random environmental effects $a(t) = r(t) + noise$, assume that $r(t)$ is nonrandom function and let the noise term be the white noise W_t which we do not know exactly what it is behavior , but only it is probability distribution, then

$$a(t) = r(t) + \alpha W_t \tag{15}$$

where α is constant , when we substitute $a(t)$ in (15) in (14) we get

$$d\,N_t = r_t\;N(t)\,dt + \alpha\;N(t)\,dW_t \tag{16}$$

Which is a geometric Brownian motion.

The geometric Brownian motion model (16) take into the consideration all the random environmental factors that effect on the population growth process so it can be regarded as more adequate models for the development for the population growth.

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